

COMPLEXES OF C -PROJECTIVE MODULES

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ABSTRACT. Inspired by a recent work of Buchweitz and Flenner, we show that, for a semidualizing bimodule C , C -perfect complexes have the ability to detect when a ring is strongly regular. It is shown that there exists a class of modules which admit minimal resolutions of C -projective modules.

Keywords: Semidualizing, C -projective, \mathcal{P}_C -resolution, C -perfect complex, strongly regular.

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1. INTRODUCTION

Let R be a left and right noetherian ring (not necessarily commutative), all modules left R -modules and C a semidualizing (R, R) -bimodule (Definition 2.1). A complex X_\bullet of R -modules is said to be C -perfect if it is quasiisomorphic to a finite complex

$$T_\bullet = 0 \longrightarrow C \otimes_R P_n \longrightarrow C \otimes_R P_{n-1} \longrightarrow \cdots \longrightarrow C \otimes_R P_1 \longrightarrow C \otimes_R P_0 \longrightarrow 0,$$

where each P_i is a finite (i.e. finitely generated) projective R -module. The *width* of such a C -perfect complex X_\bullet , denoted by $\text{wd}(X_\bullet)$, is defined to be the minimal length n of a complex T_\bullet satisfying the above conditions. Recall from [3], a ring R is called *strongly regular* whenever there exists a non-negative integer r such that every R -perfect complex is quasiisomorphic to a direct sum of R -perfect complexes of width $\leq r$. Buchweitz and Flenner, in [3], characterize the commutative noetherian rings which are strongly regular.

Our first objective is to detect when a ring is strongly regular by means of C -perfect complexes (Theorem 3.8). We also prove that C -projective modules (i.e. modules of the form $C \otimes_R P$ with P projective) have the ability to detect when a ring is hereditary (Proposition 3.1).

Our second goal is to find a class of R -modules which admit minimal resolutions of C -projective modules (see Theorem 3.10).

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2. PRELIMINARIES

Throughout, R is a left and right noetherian ring (not necessarily commutative) and let all R -modules be left R -modules. Right R -modules are identified with left modules over the opposite ring R^{op} . An (R, R) -bimodule M is both left and right R -module with compatible structures.

Definition 2.1. [9, Definition 2.1] An (R, R) -bimodule C is *semidualizing* if it is a finite R -module, finite R^{op} -module, and the following conditions hold.

- (1) The homothety map $R \xrightarrow{R\gamma} \text{Hom}_{R^{\text{op}}}(C, C)$ is an isomorphism.
- (2) The homothety map $R \xrightarrow{\gamma^R} \text{Hom}_R(C, C)$ is an isomorphism.
- (3) $\text{Ext}_R^{\geq 1}(C, C) = 0$.
- (4) $\text{Ext}_{R^{\text{op}}}^{\geq 1}(C, C) = 0$.

Assume that R is a commutative noetherian ring, then the above definition agrees with the definition of semidualizing R -module (see e.g. [9, 2.1]). Also, every finite projective R -module of rank 1 is semidualizing (see [11, Corollary 2.2.5]).

Definition 2.2. [9, Definition 3.1] A semidualizing (R, R) -bimodule C is said to be *faithfully semidualizing* if it satisfies the following conditions

- (a) If $\text{Hom}_R(C, M) = 0$, then $M = 0$ for any R -module M ;
- (b) If $\text{Hom}_{R^{\text{op}}}(C, N) = 0$, then $N = 0$ for any R^{op} -module N .

Note that over a commutative noetherian ring, all semidualizing modules are faithfully semidualizing, by [9, Proposition 3.1].

For the remainder of this section C denotes a semidualizing (R, R) -bimodule. The following class of modules is already appeared in, for example, [8], [9], and [13].

Definition 2.3. An R -module is called *C -projective* if it has the form $C \otimes_R P$ for some projective R -module P . The class of (resp. finite) C -projective modules is denoted by \mathcal{P}_C (resp. \mathcal{P}_C^f).

2.4. A complex A of R -modules is called $\text{Hom}_R(\mathcal{P}_C, -)$ -exact if $\text{Hom}_R(C \otimes_R P, A)$ is exact for each projective R -module P . The term $\text{Hom}_R(-, \mathcal{P}_C)$ -exact is defined dually.

For the notations in the next fact one may see [12, Definitions 1.4 and 1.5]

2.5. A \mathcal{P}_C -resolution of an R -module M is a complex X in \mathcal{P}_C with $X_{-n} = 0 = H_n(X)$ for all $n > 0$ and $M \cong H_0(X)$. The following exact sequence is the *augmented \mathcal{P}_C -resolution* of M associated to X :

$$X^+ = \cdots \xrightarrow{\partial_2^X} C \otimes_R P_1 \xrightarrow{\partial_1^X} C \otimes_R P_0 \longrightarrow M \longrightarrow 0.$$

A \mathcal{P}_C -resolution X of M is called *proper* if in addition X^+ is $\text{Hom}_R(\mathcal{P}_C, -)$ -exact.

The \mathcal{P}_C -projective dimension of M is the quantity

$$\mathcal{P}_C\text{-pd}(M) = \inf\{\sup\{n \geq 0 \mid X_n \neq 0\} \mid X \text{ is an } \mathcal{P}_C\text{-resolution of } M\}.$$

The objects of \mathcal{P}_C -projective dimension 0 are exactly C -projective R -modules.

The notion (*proper*) \mathcal{P}_C -coresolution is defined dually. The *augmented* \mathcal{P}_C -coresolution associated to a \mathcal{P}_C -coresolution Y is denoted by ${}^+Y$.

In [13], the authors proved the following proposition for a commutative ring R . However, by an easy inspection, one can see that it is true even if R is non-commutative.

Proposition 2.6. *Assume that C is a faithfully semidualizing (R, R) -bimodule and that M is an R -module. The following statements hold true.*

- (a) [13, Corollary 2.10(a)] *The inequality $\mathcal{P}_C\text{-pd}(M) \leq n$ holds if and only if there is a complex*

$$0 \longrightarrow C \otimes_R P_n \longrightarrow \cdots \longrightarrow C \otimes_R P_0 \longrightarrow M \longrightarrow 0$$

which is $\text{Hom}_R(\mathcal{P}_C, -)$ -exact.

- (b) [13, Theorem 2.11(a)] $\text{pd}_R(M) = \mathcal{P}_C\text{-pd}_R(C \otimes_R M)$.
(c) [13, Theorem 2.11(c)] $\mathcal{P}_C\text{-pd}_R(M) = \text{pd}_R(\text{Hom}_R(C, M))$.

Remark 2.7. By [9, Proposition 5.3] the class \mathcal{P}_C is precovering, that is, for an R -module M , there exists a projective R -module P and a homomorphism $\phi : C \otimes_R P \rightarrow M$ such that, for every projective Q , the induced map

$$\text{Hom}_R(C \otimes_R Q, C \otimes_R P) \xrightarrow{\text{Hom}_R(C \otimes_R Q, \phi)} \text{Hom}_R(C \otimes_R Q, M)$$

is surjective. Then one can iteratively take precovers to construct a complex

$$(2.7.1) \quad W = \cdots \xrightarrow{\partial_2^X} C \otimes_R P_1 \xrightarrow{\partial_1^X} C \otimes_R P_0 \longrightarrow 0$$

such that W^+ is $\text{Hom}_R(\mathcal{P}_C, -)$ -exact, where

$$W^+ = \cdots \xrightarrow{\partial_2^X} C \otimes_R P_1 \xrightarrow{\partial_1^X} C \otimes_R P_0 \xrightarrow{\phi} M \longrightarrow 0.$$

For the notions precovering, covering, preenveloping and enveloping one can see [6].

Note that if C is faithfully semidualizing (R, R) -bimodule and M is an R -module, then, by Proposition 2.6(a), $\mathcal{P}_C\text{-pd}(M)$ is equal to the length of the shortest complex as (2.7.1). Thus for any R -module M , the quantity \mathcal{P}_C -projective dimension of M , defined in [9] and [13], is equal to $\mathcal{P}_C\text{-pd}(M)$ in 2.5.

3. RESULTS

A ring R is (left) hereditary if every left ideal is projective. The Cartan-Eilenberg theorem [10, Theorem 4.19] shows that R is hereditary if and only if every submodule of a projective module is projective. We show that the quality of being hereditary can be detected by C -projective modules, which is interesting on its own.

Proposition 3.1. *Assume that C runs through the class of faithfully semidualizing (R, R) -bimodules. The following statements are equivalent.*

- (i) R is left hereditary.
- (ii) For any C , every submodule of a C -projective R -module is also C -projective.
- (iii) There exists a C such that every submodule of a C -projective R -module is also C -projective.

Proof. (i) \Rightarrow (ii). Let C be a faithfully semidualizing bimodule and N a submodule of $C \otimes_R P$, where P is a projective R -module. Then one gets the exact sequence $0 \rightarrow \text{Hom}_R(C, N) \rightarrow P$. As R is left hereditary, $\text{Hom}_R(C, N)$ is a projective R -module. By Proposition 2.6(c), $\mathcal{P}_C\text{-pd}(N) = \text{pd}(\text{Hom}_R(C, N)) = 0$.

(ii) \Rightarrow (iii) is immediate.

(iii) \Rightarrow (i). As every submodule of a C -projective R -module is C -projective, for any R -module M one has $\mathcal{P}_C\text{-pd}(M) \leq 1$. Then for any R -module N one gets $\text{pd}(N) = \mathcal{P}_C\text{-pd}(C \otimes_R N) \leq 1$, by Proposition 2.6(b). It follows that every submodule of a projective is projective and so, by [10, Theorem 4.19], R is left hereditary. \square

Definition 3.2. A complex X_\bullet of R -modules is called C -perfect if it is quasiisomorphic to a finite complex

$$T_\bullet = 0 \rightarrow C \otimes_R P_n \rightarrow C \otimes_R P_{n-1} \rightarrow \cdots \rightarrow C \otimes_R P_1 \rightarrow C \otimes_R P_0 \rightarrow 0,$$

where P_i are finite projective R -modules. The *width* of such a C -perfect complex X_\bullet , denoted by $\text{wd}(X_\bullet)$, is defined to be the minimal length n of a complex T_\bullet satisfying the above conditions. A C -perfect complex X_\bullet is called *indecomposable* if it is not quasiisomorphic to a direct sum of two non-trivial C -perfect complexes.

Definition 3.3. [3, Definition 1.1] A ring R is called *strongly r -regular* if every perfect complex over R is quasiisomorphic to a direct sum of perfect complexes of width $\leq r$. If R is strongly r -regular for some r then it will be called *strongly regular*.

Remark 3.4. As Professor Ragnar-Olaf Buchweitz kindly pointed out in his personal communication with the authors, in [3] it should be added the blanket statement that rings are noetherian and modules are finite. Thus Definition 3.3 agrees with [3, Definition 1.1]. Indeed, over a noetherian ring every perfect complex has bounded and finite homology.

Note that a hereditary ring R is strongly 1-regular, see [3, Remark 1.2].

In order to bring the results Theorem 3.8 and Proposition 3.9, we quote some preliminaries.

Definition 3.5. [7, III.3.2(b)] and [4, Definition 2.2.8] Let $\alpha : A \rightarrow B$ be a morphism of R -complexes. The *mapping cone* of α , $\text{Cone}(\alpha)$, is a complex which is given by

$$(\text{Cone}(\alpha))_n = B_n \oplus A_{n-1} \quad \text{and} \quad \partial_n^{\text{Cone}(\alpha)} = \begin{pmatrix} \partial_n^B & \alpha_{n-1} \\ 0 & -\partial_{n-1}^A \end{pmatrix}.$$

It easy to see that the following lemma is also true if R is non-commutative.

Lemma 3.6. *Let $\alpha : A \rightarrow B$ be a morphism of R -complexes and M be an R -module. The following statements hold true.*

- (a) [4, Lemma 2.2.10] *The morphism α is a quasiisomorphism if and only if $\text{Cone}(\alpha)$ is acyclic.*
- (b) [4, Lemma 2.3.11] $\text{Cone}(\text{Hom}_R(M, \alpha)) \cong \text{Hom}_R(M, \text{Cone}(\alpha))$.
- (c) [4, Lemma 2.4.11] $\text{Cone}(M \otimes_R \alpha) \cong M \otimes_R \text{Cone}(\alpha)$.

Remark 3.7. Let C be a semidualizing (R, R) -bimodule. Assume that $X = 0 \rightarrow X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow 0$ is an exact complex of R -modules.

- (a) If each X_i is a projective R -module, then it is easy to see that the induced complex $C \otimes_R X$ is exact.
- (b) If each X_i is a C -projective R -module, then the induced complex $\text{Hom}_R(C, X)$ is exact, since $\text{Ext}_R^{\geq 1}(C, X_i) = 0$.

Theorem 3.8. *The following statements are equivalent.*

- (i) R is strongly r -regular.
- (ii) *For any faithfully semidualizing bimodule C , every C -perfect complex is quasiisomorphic to a direct sum of C -perfect complexes of width $\leq r$.*
- (iii) *There exists a faithfully semidualizing bimodule C such that every C -perfect complex is quasiisomorphic to a direct sum of C -perfect complexes of width $\leq r$.*

Proof. (i) \Rightarrow (ii). Let R be strongly r -regular, C a faithfully semidualizing bimodule. Assume that X_\bullet is a C -perfect complex. Then, by Definition 3.2, there exists a finite complex

$$T_\bullet = 0 \rightarrow C \otimes_R P_n \rightarrow C \otimes_R P_{n-1} \rightarrow \cdots \rightarrow C \otimes_R P_0 \rightarrow 0,$$

such that each P_i is a finite projective R -module and X_\bullet is quasiisomorphic to T_\bullet . Therefore $\text{Hom}_R(C, T_\bullet) \cong 0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow 0$ is a perfect complex. By Definition 3.3, there is a quasiisomorphism $\alpha : \text{Hom}_R(C, T_\bullet) \xrightarrow{\cong} \bigoplus_{i=1}^s F_\bullet^{(i)}$,

where each $F_{\bullet}^{(i)}$ is a perfect complex of width $\leq r$. We may assume that each $F_{\bullet}^{(i)}$ is a finite complex of finite projective R -modules. By Lemma 3.6(a), $\text{Cone}(\alpha)$ is acyclic. As $\text{Cone}(\alpha)$ is a finite complex of projective R -modules, Remark 3.7 implies that the complex $C \otimes_R \text{Cone}(\alpha)$ is acyclic. By Lemma 3.6, the complex $\text{Cone}(C \otimes_R \alpha)$ is acyclic too and so $C \otimes_R \alpha$ is quasiisomorphism. Therefore T_{\bullet} is quasiisomorphic to $\bigoplus_{i=1}^s C \otimes_R F_{\bullet}^{(i)}$. Note that each $C \otimes_R F_{\bullet}^{(i)}$ is a C -perfect complex of width $\leq r$.

(ii) \Rightarrow (iii) is immediate.

(iii) \Rightarrow (i). Let Y_{\bullet} be a perfect complex. Then, by Definition 3.2, there is a finite complex $F_{\bullet} = 0 \rightarrow P_m \rightarrow P_{m-1} \rightarrow \cdots \rightarrow P_0 \rightarrow 0$ of finite projective modules which is quasiisomorphic to Y_{\bullet} . As $C \otimes_R F_{\bullet}$ is a C -perfect complex, our assumption implies that there is a quasiisomorphism $\beta : C \otimes_R F_{\bullet} \xrightarrow{\sim} \bigoplus_{i=1}^s T_{\bullet}^{(i)}$, where each $T_{\bullet}^{(i)}$ is a C -perfect complex of width $\leq r$. We may assume that, for each i ,

$$T_{\bullet}^{(i)} = 0 \rightarrow C \otimes_R P_{n_i}^{(i)} \rightarrow \cdots \rightarrow C \otimes_R P_0^{(i)} \rightarrow 0$$

where each $P_j^{(i)}$ is a finite projective R -module. Similar to the proof of (i) \Rightarrow (ii), one observes that $\text{Hom}_R(C, \beta)$ is a quasiisomorphism. Therefore F_{\bullet} is quasiisomorphic to $\bigoplus_{i=1}^s \text{Hom}_R(C, T_{\bullet}^{(i)})$. Note that each $\text{Hom}_R(C, T_{\bullet}^{(i)})$ is a perfect complex of width $\leq r$. Thus R is strongly r -regular. \square

In [2, Section 1], Avramov and Martsinkovsky define a general notion of minimality for complexes: A complex X is *minimal* if every homotopy equivalence $\sigma : X \rightarrow X$ is an isomorphism. In [14, Lemma 4.8], it is proved that, over a commutative local ring R with maximal ideal \mathfrak{m} , a complex X consisting of modules in \mathcal{P}_C^f is minimal if and only if $\partial^X(X) \subseteq \mathfrak{m}X$.

In consistent to [3, Lemma 1.6] we prove the following proposition.

Proposition 3.9. *Let R be a commutative noetherian local ring, C a semidualizing R -module. The following statements hold true.*

(a) *Every C -perfect complex X_{\bullet} is quasiisomorphic to a minimal finite complex*

$$T_{\bullet} = 0 \rightarrow C \otimes_R F_n \rightarrow C \otimes_R F_{n-1} \rightarrow \cdots \rightarrow C \otimes_R F_1 \rightarrow C \otimes_R F_0 \rightarrow 0,$$

where each F_i is finite free R -module.

(b) *If two minimal finite complexes of modules of the form $C^m = \bigoplus^m C$ are quasiisomorphic, then they are isomorphic.*

Proof. (a). By Definition 3.2, a C -perfect complex X_{\bullet} is quasiisomorphic to a finite complex

$$T_{\bullet} = 0 \rightarrow C \otimes_R P_n \rightarrow C \otimes_R P_{n-1} \rightarrow \cdots \rightarrow C \otimes_R P_1 \rightarrow C \otimes_R P_0 \rightarrow 0,$$

where each P_i is a finite free R -module. The complex $\text{Hom}_R(C, T_{\bullet})$ is a perfect complex and so, by [3, Lemma 1.6(1)], there exist a minimal finite complex F_{\bullet} of

finite free R -modules and a quasiisomorphism $\alpha : \text{Hom}_R(C, T_\bullet) \xrightarrow{\sim} F_\bullet$. As in the proof of Theorem 3.8, it follows that $C \otimes_R \alpha : C \otimes_R \text{Hom}_R(C, T_\bullet) \rightarrow C \otimes_R F_\bullet$ is a quasiisomorphism. As $C \otimes_R F_\bullet$ is a minimal finite complex, we are done.

(b). Let T_\bullet and L_\bullet be two minimal finite complexes of modules of the form C^m . Assume that $\alpha : T_\bullet \rightarrow L_\bullet$ is a quasiisomorphism. Then, by Remark 3.7 and Lemma 3.6, $\text{Hom}_R(C, \alpha) : \text{Hom}_R(C, T_\bullet) \rightarrow \text{Hom}_R(C, L_\bullet)$ is a quasiisomorphism of minimal finite complexes of finite free R -modules. Thus, by the proof of [3, Lemma 1.6(2)], $\text{Hom}_R(C, \alpha)$ is an isomorphism. Now, there is a commutative diagram of complexes and morphisms

$$\begin{array}{ccc} T_\bullet & \xrightarrow[\alpha]{\simeq} & L_\bullet \\ \uparrow \cong & & \uparrow \cong \\ C \otimes_R \text{Hom}_R(C, T_\bullet) & \xrightarrow[C \otimes_R \text{Hom}_R(C, \alpha)]{\cong} & C \otimes_R \text{Hom}_R(C, L_\bullet), \end{array}$$

where the vertical morphisms are natural isomorphisms. This implies that α itself must be an isomorphism. \square

It is proved in [14, Lemma 4.9] that every finite module M over a commutative noetherian local ring R with $\mathcal{P}_C^f\text{-pd}(M) < \infty$ admits a minimal \mathcal{P}_C^f -resolution. Now we show that every finite R -module which has a proper \mathcal{P}_C -resolution, admits a minimal proper one. Note that if $\mathcal{P}_C^f\text{-pd}(M) < \infty$ then M admits a proper \mathcal{P}_C -resolution (see proof of [13, Corollary 2.10]).

Theorem 3.10. *Assume that R is a commutative noetherian local ring and that C is a semidualizing R -module. Then \mathcal{P}_C^f is covering in the category of finite R -modules. For any finite R -module M , there is a complex $X = \cdots \rightarrow C^{n_1} \rightarrow C^{n_0} \rightarrow 0$ with the following properties.*

- (1) $X^+ = \cdots \rightarrow C^{n_1} \rightarrow C^{n_0} \rightarrow M \rightarrow 0$ is $\text{Hom}_R(\mathcal{P}_C, -)$ -exact.
- (2) X is a minimal complex.

If M admits a proper \mathcal{P}_C -resolution, then X^+ is exact and so X is a minimal proper \mathcal{P}_C -resolution of M .

Proof. Let M be a finite R -module. Assume that $n_0 = \nu(\text{Hom}_R(C, M))$ denotes the number of a minimal set of generators of $\text{Hom}_R(C, M)$ and that $\alpha : R^{n_0} \rightarrow \text{Hom}_R(C, M)$ is the natural epimorphism. As α is a \mathcal{P}^f -cover of $\text{Hom}_R(C, M)$, the natural map $\beta = C \otimes_R R^{n_0} \xrightarrow{C \otimes_R \alpha} C \otimes_R \text{Hom}_R(C, M) \xrightarrow{\nu_M} M$ is a \mathcal{P}_C^f -cover of M . Set $M_1 = \text{Ker} \beta$ and $n_1 = \nu(\text{Hom}_R(C, M_1))$. Thus there is a \mathcal{P}_C^f -cover $\beta_1 : C \otimes_R R^{n_1} \rightarrow M_1$. Proceeding in this way one obtains a complex

$$X = \cdots \xrightarrow{\partial_2 = \epsilon_2 \beta_2} C \otimes_R R^{n_1} \xrightarrow{\partial_1 = \epsilon_1 \beta_1} C \otimes_R R^{n_0} \rightarrow 0,$$

where $\epsilon_i : M_i \rightarrow C \otimes_R R^{n_{i-1}}$ is the inclusion map for all $i \geq 1$. As the maps in X are obtained by \mathcal{P}_C^f -covers, the complex X^+ is $\text{Hom}_R(\mathcal{P}_C, -)$ -exact. It is easy to see that $\text{Hom}_R(C, X)$ is minimal free resolution of $\text{Hom}_R(C, M)$. Now we show that X is a minimal complex. Let $f : X \rightarrow X$ be a morphism which is homotopic to id_X . It is easy to see that the morphism $\text{Hom}_R(C, f)$ is homotopic to $\text{id}_{\text{Hom}_R(C, X)}$. As the complex $\text{Hom}_R(C, X)$ is minimal, by [2, Proposition 1.7], the morphism $\text{Hom}_R(C, f)$ is an isomorphism. The commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow \cong & & \downarrow \cong \\ C \otimes_R \text{Hom}_R(C, X) & \xrightarrow[\text{Hom}_R(C, f)]{\cong} & C \otimes_R \text{Hom}_R(C, X), \end{array}$$

with vertical natural isomorphisms, implies that f is an isomorphism. Therefore, by [2, Proposition 1.7], X is minimal. If M admits a proper \mathcal{P}_C -resolution, then by [13, Corollary 2.3], X^+ is exact. \square

The proof of the next lemma is similar to [13, Corollary 2.3].

Lemma 3.11. *Let R be a commutative noetherian ring and let M be a finite R -module. Assume that C is a semidualizing R -module. The following are equivalent.*

- (i) *M admits a proper \mathcal{P}_C^f -coresolution.*
- (ii) *Every $\text{Hom}_R(-, \mathcal{P}_C^f)$ -exact complex of the form*

$$0 \longrightarrow M \longrightarrow C \otimes_R Q_0 \longrightarrow C \otimes_R Q_{-1} \longrightarrow \cdots$$

is exact, where Q_i is an object of \mathcal{P}^f for all $i \leq 0$.

- (iii) *The natural homomorphism $M \rightarrow \text{Hom}_R(\text{Hom}_R(M, C), C)$ is an isomorphism and $\text{Ext}_R^{\geq 1}(\text{Hom}_R(M, C), C) = 0$.*

Proposition 3.12. *Assume that R is a commutative noetherian local ring and that C is a semidualizing R -module. Then \mathcal{P}_C^f is enveloping in the category of finite R -modules. For any finite R -module M , there is a complex $Y = 0 \rightarrow C^{m_0} \rightarrow C^{m_1} \rightarrow \cdots$ with the following properties.*

- (1) *${}^+Y = 0 \rightarrow M \rightarrow C^{m_0} \rightarrow C^{m_1} \rightarrow \cdots$ is $\text{Hom}_R(-, \mathcal{P}_C)$ -exact.*
- (2) *Y is a minimal complex.*

If M admits a proper \mathcal{P}_C^f -coresolution, then ${}^+Y$ is exact and so Y is a minimal proper \mathcal{P}_C -coresolution of M .

Proof. Let M be a finite R -module. Assume that $m_0 = \nu(\text{Hom}_R(M, C))$ denotes the number of a minimal set of generators of $\text{Hom}_R(M, C)$ and that $\alpha : R^{m_0} \rightarrow \text{Hom}_R(M, C)$ is the natural \mathcal{P}^f -cover of $\text{Hom}_R(M, C)$. It follows that $\gamma = M \xrightarrow{\delta_M} \text{Hom}_R(\text{Hom}_R(M, C), C) \xrightarrow{\text{Hom}_R(\alpha, C)} \text{Hom}_R(R^{m_0}, C)$ is a \mathcal{P}_C^f -envelope of

M . Set $M_{-1} = \text{Coker } \gamma$ and $m_1 = \nu(\text{Hom}_R(M_{-1}, C))$. As mentioned, there is a \mathcal{P}_C^f -envelope $\gamma_1 : M_{-1} \rightarrow \text{Hom}_R(R^{m_1}, C)$. Proceeding in this way one obtains a complex $Y = 0 \rightarrow \text{Hom}_R(R^{m_0}, C) \xrightarrow{\partial_0 = \gamma_1 \pi_1} \text{Hom}_R(R^{m_1}, C) \xrightarrow{\partial_{-1} = \gamma_2 \pi_2} \cdots$, where π_i is the natural epimorphism for all $i \geq 1$. Since the maps in Y are obtained by \mathcal{P}_C^f -envelopes, the complex ${}^+Y$ is $\text{Hom}_R(-, \mathcal{P}_C)$ -exact. It is easy to see that $\text{Hom}_R(Y, C)$ is minimal free resolution of $\text{Hom}_R(M, C)$. Similar to the proof of Theorem 3.10, we find that Y is a minimal complex. If M admits a proper \mathcal{P}_C^f -coresolution, then, by Lemma 3.11, ${}^+Y$ is exact. \square

In the following example we find an R -module M with $\mathcal{P}_C\text{-pd}(M) = \infty$ which admits a minimal proper \mathcal{P}_C -resolution. This example shows that a commutative noetherian local ring which admits an exact zero-divisor is not a strongly regular ring.

Example 3.13. Let R be a commutative noetherian local ring, C a semidualizing R -module. Assume that x, y form a pair of exact zero-divisors on both R and C (e.g. see [1, Example 3.2]). Then $\mathcal{P}_C\text{-pd}(C/xC) = \text{pd}(R/xR) = \infty$. The complex

$$T_\bullet = \cdots \xrightarrow{x} C \xrightarrow{y} C \xrightarrow{x} C \rightarrow 0 \quad (\text{resp. } L_\bullet = 0 \rightarrow C \xrightarrow{x} C \xrightarrow{y} C \xrightarrow{x} \cdots)$$

is a minimal \mathcal{P}_C -resolution (resp. \mathcal{P}_C -coresolution) of C/xC . By [1, Proposition 3.4], C/xC is a semidualizing R/xR -module. By [5, Proposition 2.13], there are isomorphisms

$$\text{Hom}_R(C, C/xC) \cong \text{Hom}_{R/xR}(C/xC, C/xC) \cong R/xR,$$

$$\text{Hom}_R(C/xC, C) \cong \text{Hom}_{R/xR}(C/xC, C/xC) \cong R/xR.$$

Applying $\text{Hom}_R(C, -)$ and $\text{Hom}_R(-, C)$ on the above complexes, respectively, would result the isomorphisms $\text{Hom}_R(C, T_\bullet^+) \cong F_\bullet^+$ and $\text{Hom}_R({}^+L_\bullet, C) \cong F_\bullet^+$, where F_\bullet^+ is the exact complex $\cdots \xrightarrow{y} R \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \rightarrow R/xR \rightarrow 0$. Therefore T_\bullet (resp. L_\bullet) is a minimal proper \mathcal{P}_C -resolution (resp. \mathcal{P}_C -coresolution) of C/xC .

For each n , one obtains a C -perfect complex of length n as

$$T_\bullet^{(n)} = 0 \rightarrow C \rightarrow C \rightarrow \cdots \xrightarrow{x} C \xrightarrow{y} C \xrightarrow{x} C \rightarrow 0,$$

where $T_i^{(n)} = T_i$ for all $0 \leq i \leq n$ and $T_i^{(n)} = 0$ otherwise. Note that the induced map $\bar{d}_i : T_i^{(n)} / \text{Ker } d_i \rightarrow T_{i-1}^{(n)}$ is injective, where $\text{Ker } d_i$ is equal to yC or xC . As C is indecomposable R -module, $T_\bullet^{(n)}$ is indecomposable which has a similar proof to [3, Proposition 1.5].

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